# 10/31 Recitation Solutions

1. Geometrically determine the eigenvalues of the matrices associated to the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by:

   (a) the identity transformation ($T(\mathbf{v}) = \mathbf{v}$)

   Let $A$ be the associated matrix. Since $Av = v$ for all $\mathbf{v} \in \mathbb{R}^2$, we have that every nonzero vector is an eigenvector corresponding to the eigenvalue $\lambda = 1$. Since there are no other eigenvectors, $\lambda = 1$ is the only eigenvalue.

   (b) projection of vectors onto the line $L$ given by $y = x$.

   Let $A$ be the associated matrix. If $\mathbf{v} \in L$, then $A\mathbf{v} = \mathbf{v}$ and so $\lambda = 1$ is an eigenvalue. If $\mathbf{v} \in L^\perp$, then $A\mathbf{v} = \mathbf{0} = 0\mathbf{v}$ and so $\lambda = 0$ is also an eigenvalue. If $\mathbf{v}$ is not in $L$ or $L^\perp$, then since $A\mathbf{v} \in L$, it is not a multiple of $\mathbf{v}$ and so there are no other eigenvalues.

   (c) the trivial transformation ($T(\mathbf{v}) = \mathbf{0}$) for all $\mathbf{v} \in \mathbb{R}^2$.

   Let $A$ be the associated matrix. Since $Av = \mathbf{0} = 0\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^2$, we have that every nonzero vector is an eigenvector corresponding to the eigenvalue $\lambda = 0$. Since there are no other eigenvectors, $\lambda = 0$ is the only eigenvalue.

2. Let $A$ be an invertible matrix and let $\lambda$ be an eigenvalue of $A$. Show that $\frac{1}{\lambda}$ is an eigenvalue of $A^{-1}$.

   Since $\lambda$ is an eigenvalue, we have $A\mathbf{v} = \lambda\mathbf{v}$ for some nonzero vector $\mathbf{v}$. Multiplying both sides of $A^{-1}$ yields

$$\begin{aligned} A^{-1}(A\mathbf{v}) &= A^{-1}(\lambda\mathbf{v}) \\ (A^{-1}A)\mathbf{v} &= \lambda A^{-1}\mathbf{v} \\ \mathbf{v} &= \lambda A^{-1}\mathbf{v} \end{aligned}$$

   Since $\mathbf{v}$ is nonzero, the last equality implies that $\lambda \neq 0$; hence we may divide the equation by $\lambda$ to get

$$A^{-1}\mathbf{v} = \frac{1}{\lambda}\mathbf{v};$$

   that is, $\frac{1}{\lambda}$ is an eigenvalue of $A^{-1}$.

3. The matrix $M = \begin{bmatrix} .95 & .1 & 0 \\ .05 & .9 & .5 \\ 0 & 0 & .5 \end{bmatrix}$ describes the population migration between three states: A, B, and C. We assume it is a closed system: nobody is born, nobody dies, and nobody moves in or out of these states. Note that the first column says that in any given year, of the people living in A, 95% of people stay in A, 5% of residents move to B, and 0% of residents move to C.

   Suppose 50% of the total population of these three states lives in A, 40% lives in B, and 10% lives in C. Let $\mathbf{x}_0$ be the associated initial state vector. In this problem, you will determine the populations of each state, in the long run. Complete "Worksheet Problem 1" in the **Julia11EigenvaluesEigenvectors** notebook.

   (a) Let $\mathbf{x}_k = M^k\mathbf{x}_0$. Use a for loop to compute $\mathbf{x}_k$ for all $1 \leq k \leq 30$. Does the Markov chain appear to converge? If so to what?

   **Code:**
   ```
   M = [.95 .1 0; .05 .9 .5; 0 0 .5]
   x0 = [.5; .4; .1]
   for i = 1:30
   println(M^i*x0)
   end
   ```

   These vectors definitely converge.

   (b) Use Julia to find the eigenvalues and eigenvectors of $M$. Express $\mathbf{x}_0$ as a linear combination of the eigenvectors (use Julia to solve an appropriate system).

   This first bit of code provides the eigenvalues and eigenvectors:

   **Code:**
   ```
   using LinearAlgebra
   valM,vecM = eigen(M)
   ```

   This next bit solves a system that provides the weights needed (in the vector c below) to express $\mathbf{x}_0$ as a linear combination of the eigenvectors

   **Code:**
   ```
   A = [vecM[:,1] vecM[:,2] vecM[:,3]]
   c = A\x0
   ```

   We have $\mathbf{x}_0 = A\mathbf{c}$.

(c) Write down a formula for $\mathbf{x}_k = M^k\mathbf{x}_0$ (by hand) and determine the limit as $k \to \infty$

Using the Julia code notation, we have

$$\mathbf{x}_0 = A\mathbf{c} = c_1\text{vecM[:,1]} + c_1\text{vecM[:,2]} + c_1\text{vecM[:,3]}$$

where $c_1, c_2, c_3$ are the entries of $\mathbf{c}$. Hence

$$\mathbf{x}_k = M^k\mathbf{x}_0 = c_1M^k\text{vecM[:,1]} + c_2M^k\text{vecM[:,2]} + c_3M^k\text{vecM[:,3]}.$$

Looking back at the code output, the eigenvalues corresponding to the three eigenvectors are $.5, .85, 1$, respectively. Now recall that if $M\mathbf{v} = \lambda\mathbf{v}$, then $M^k\mathbf{v} = \lambda^k\mathbf{v}$. Thus

Hence

$$\mathbf{x}_k = c_1(.5)^k\text{vecM[:,1]} + c_2(.85)^k\text{vecM[:,2]} + c_3(1)^k\text{vecM[:,3]}.$$

Now as $k \to \infty$, we have $\mathbf{x}_k \to c_3\text{vecM[:,3]}$. Using Julia, this value is computed by:
**Code:**
c[3]*vecM[:,3]

This gives the vector $\begin{bmatrix} 2/3 \\ 1/3 \\ 0 \end{bmatrix}$.

(d) In the long run, what percentage of the population do you expect to live in each state?

You expect about 66.67% to live in A, 33.34% to live in B, and 0% to live in C.

(e) Show that the vector that the Markov chain approached is an eigenvector corresponding to the eigenvalue 1.

We need to show $M \begin{bmatrix} 2/3 \\ 1/3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 0 \end{bmatrix}$. We can do this by hand, or using the following code
**Code:**
M*[2/3 ; 1/3 ; 0]

and verifying the output is as desired.

4. Suppose the weather on a given day depends only on the weather the previous day. If the weather is sunny one day, then there is a 90% chance that it is sunny the next day and a

10% chance that it is rainy the next day. On the other hand if the weather is rainy, then there is a 50% chance that it is sunny the next day and a 50% chance that it is rainy the next day. Complete "Worksheet Problem 2" in the **Julia11EigenvaluesEigenvectors** notebook.

(a) Write down a Markov matrix $M$ encoding these probabilities

$$M = \begin{bmatrix} .9 & .5 \\ .1 & .5 \end{bmatrix}$$

(b) Choose any initial state vector $\mathbf{x}_0$ and use Julia to compute $\mathbf{x}_k$ for all $1 \leq k \leq 30$. To what vector does it converge? Repeat with anothe inital vector.

Choose $\mathbf{x}_0 = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$ and use the following code:

**Code:**

```
M = [.9 .5; .1 .5]
x = [.5 ; .5]
for i = 1:30
println(M^i*x)
end
```

We see that the Markov chain converges to $\begin{bmatrix} .833333333... \\ .16666666666... \end{bmatrix}$ or $\begin{bmatrix} 5/6 \\ 1/6 \end{bmatrix}$.

Choosing any other initial value converges to the same vector.

(c) Show that the vector you found in b is an eigenvector corresponding to the eigenvalue of 1.

We need to show $M \begin{bmatrix} 5/6 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 5/6 \\ 1/6 \end{bmatrix}$. We can do this by hand, or using the following code

**Code:**

```
M*[5/6; 1/6]
```

and verifying the output is as desired.

5. **Looking Ahead** After completing the above two problems, you will have observed something special about Markov chains. Those examples converged to an eigenvector corresponding to the eigenvalue 1 (this is called a *steady state vector*). We'll see why (and under which conditions) this happens next week.